Eigenvectors and Eigenvalues Cheat Sheet

Let's start by defining 3 vectors of length/dimension n: $\vec{u}$, $\vec{v}$, and $\vec{w}$:

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

We call this collection of vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ a **set** of vectors, denoted as $\{\vec{u}, \vec{v}, \vec{w}\}$

A set of vectors is also called a **vector space**

If a vector space is completely contained in another vector space, we call it a **subspace**

$\{\vec{u}, \vec{v}\}$ is a subspace of $\{\vec{u}, \vec{v}, \vec{w}\}$

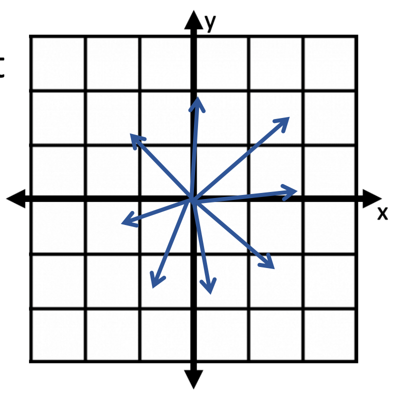
Mathematically: $\{\vec{u}, \vec{v}\} \subset \{\vec{u}, \vec{v}, \vec{w}\}$

We can create a **linear combination** of $\{\vec{u}, \vec{v}, \vec{w}\}$ by multiplying them by scalars α, β, and γ and adding them together:

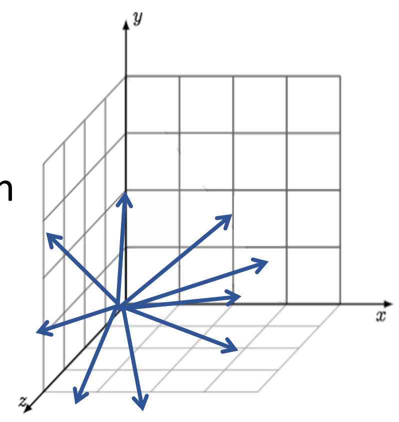
$$\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ \vdots & \vdots & \vdots \\ u_n & v_n & w_n \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \alpha \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \beta \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \gamma \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$= (\vec{u}\ \vec{v}\ \vec{w}) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \alpha\vec{u} + \beta\vec{v} + \gamma\vec{w}$$

The **vector space** that includes all real vectors of dimension n=2 is called $\mathbb{R}^2$ and can be plotted on a 2D grid



The **vector space** that includes all real vectors of dimension n=3 is called $\mathbb{R}^3$ and can be plotted on a 3D grid



If we take $\{\vec{u}, \vec{v}, \vec{w}\}$ for n=3, we get a set in $\mathbb{R}^3$:

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

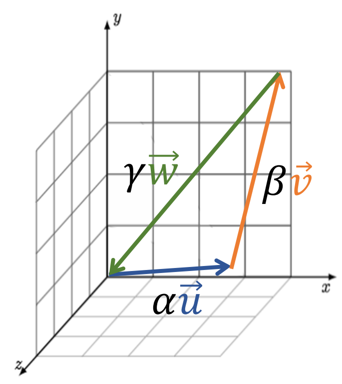
This vector space is a subspace of $\mathbb{R}^3$

If we take $\{\vec{u}, \vec{v}, \vec{w}\}$ for n=2, we get a set in $\mathbb{R}^2$:

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

This vector space is a subspace of $\mathbb{R}^2$ and a subspace of $\mathbb{R}^3$

A set of vectors is **linearly dependent** if one vector is equal to a linear combination of the other vectors



$\{\vec{u}, \vec{v}, \vec{w}\}$ is **linearly dependent** iff:

$$(\vec{u}\ \vec{v}\ \vec{w}) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = 0$$

where α, β, and γ must NOT all be zero (also called the "trivial" solution)

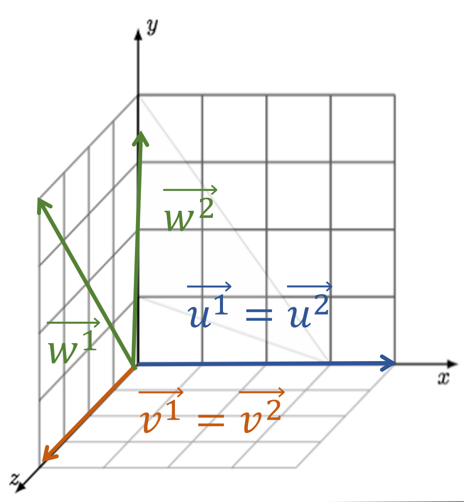
If a set of vectors is *not* linearly dependent, it is **linearly independent**

All sets of orthogonal vectors are linearly independent

A vector space of dimension n can have exactly n linearly independent vectors in a set

If you can create all vectors in a vector space V using a linear combination of $\{\vec{u}, \vec{v}, \vec{w}\}$:
V is **spanned** by $\{\vec{u}, \vec{v}, \vec{w}\}$

If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent and spans vector space V:
$\{\vec{u}, \vec{v}, \vec{w}\}$ a **basis** of V



$$\{\vec{u^1}, \vec{v^1}, \vec{w^1}\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$ spans $\mathbb{R}^3$ and is linearly dependent

→ not a basis

$$\{\vec{u^2}, \vec{v^2}, \vec{w^2}\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$ spans $\mathbb{R}^3$ and is linearly independent

→ is a basis

Quantum superpositions are the same as linear combinations of the qubit energy states / basis

$$|\psi\rangle = (|0\rangle, |1\rangle) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|0\rangle + \beta|1\rangle$$

We can create another qubit basis with equal superpositions of the energy states:

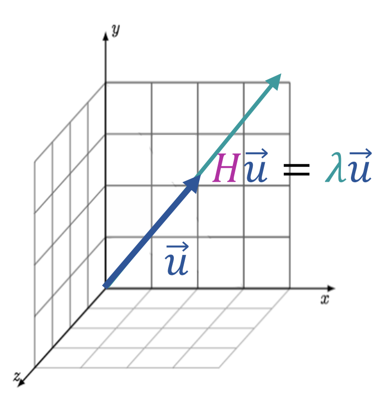
$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \quad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Eigenvalues and eigenvectors are sets of scalars (values) and vectors characteristic to a particular matrix and which satisfy:

$$H\vec{u} = \lambda\vec{u}$$

Where H is a matrix, λ is a scalar, and $\vec{u}$ is a vector

An eigenvector $\vec{u}$ does NOT change direction when multiplied by its matrix H, but it is scaled by a factor λ



If multiple eigenvectors correspond to a single eigenvalue, the eigenvectors are linearly dependent

An eigenbasis is a basis that is made up of eigenvectors of a matrix

Eigenvectors and eigenvalues are defined *relative* to a particular matrix

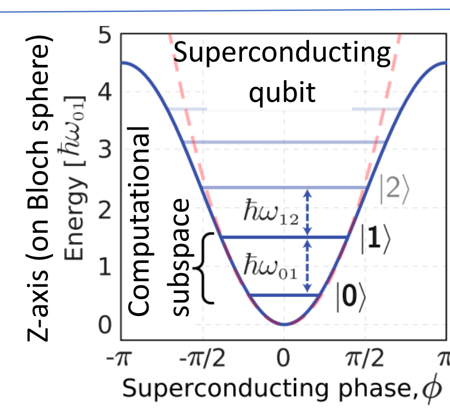
i.e. an eigenvector for one matrix may not be an eigenvector for a different matrix

The $|0\rangle$ and $|1\rangle$ quantum states we have been discussing all semester are eigenvectors of our quantum system's Hamiltonian matrix (!!!)

The time-independent Schrodinger equation is the same as the eigenvalue/eigenvector equation:

$$H\vec{\Psi} = E\vec{\Psi}$$

→ the energies of these quantum states are equal to their eigenvalues



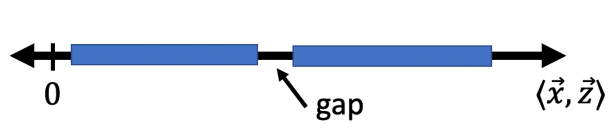
Modified from Krantz et. al., Appl. Phys. Rev., 2019.

You can see there are more (actually an infinite number of) eigenvalues of higher energy, but we only care about the subspace spanned by $\{|0\rangle, |1\rangle\}$, which we call the **computational subspace**

A **Hilbert space** is a type of vector space that has special properties that make it easy to define lengths and angles of its vectors (the inner product) and to perform calculus

All finite-dimensional ($n \neq \infty$) vector spaces that have a meaningful inner product are Hilbert spaces

Infinite-dimensional vector spaces have an additional constraint that they are "complete" – meaning (informally) that there are no gaps in the set of possible inner products



If $\{\vec{x}, \vec{z}\} \in V$, then V is NOT a Hilbert space

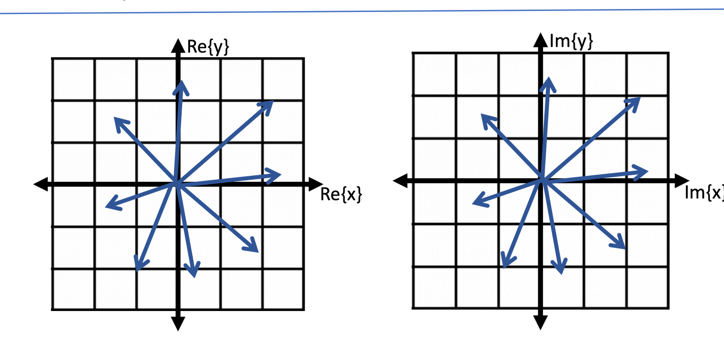
All vector spaces that can be mapped onto $\mathbb{R}^n$ (including $\mathbb{C}^n$) are Hilbert spaces

You can *always* assume you are in a Hilbert space in this course.

A single qubit can be described by a **vector space in** $\mathbb{C}^2$ where $\mathbb{C}^2$ contains all 2-dimensional complex vectors

$\mathbb{C}^2$ is a Hilbert space

We can represent $\mathbb{C}^2$ with two 2D graphs – one showing the real part of the vectors and one showing the imaginary part of the vectors



$$\sigma_x|\psi_{x+}\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1\,|+\rangle$$

$$\sigma_y|\psi_{y+}\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = 1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$\sigma_z|\psi_{z+}\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1|0\rangle$$

$$\sigma_x|\psi_{x-}\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = -1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = -1\,|-\rangle$$

$$\sigma_y|\psi_{y-}\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = -1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

$$\sigma_z|\psi_{z-}\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1|1\rangle$$

$\{|\psi_{x+}\rangle, |\psi_{x-}\rangle\}$,
$\{|\psi_{y+}\rangle, |\psi_{y-}\rangle\}$,
$\{|\psi_{z+}\rangle, |\psi_{z-}\rangle\}$:

- are orthogonal and thus linearly independent
- span $\mathbb{C}^2$
- are normalized

$\{|+\rangle, |-\rangle\}$ are eigenvectors of σ_x
→ are an orthonormal eigenbasis of σ_x

$\{|0\rangle, |1\rangle\}$ are eigenvectors of σ_z
→ are an orthonormal eigenbasis of σ_z

$\{|0\rangle, |1\rangle\}$ are eigenvectors of both H and σ_z

Their eigenvalues correspond to the qubit energy levels

This is why we measure in the Z-basis (also called the energy basis)