Optional Winter Review

This winter review sheet is meant to be an overview of many of the topics we covered in the first semester of the Qubit by Qubit quantum course. These are all of the skills that we will want to be familiar with as we go deeper into learning about quantum mechanics and quantum algorithms in semester 2. Most of the problems are based on past homework problems. We encourage you to refer to past homework problems and lab recordings if you would like to review these concepts. Solutions are at the end of the sheet. Students should work independently and check their solutions. There is no assignment to be submitted on Canvas.

We have also included some of the challenge problems from the homework that are particularly important in quantum computing. We encourage students to try them, or read through their solutions if they want a good challenge!

Topic 1: Bits, Booleans and Gates

1) Convert the following binary numbers to base 10:

   a) 10_2

   b) 111_2

   c) 1001_2

   d) 11001_2

2) Convert the following base 10 numbers to binary:

   a) 5_{10}

   b) 23_{10}

   c) 127_{10}

3) Add the following binary numbers:

a) $11_2 + 11_2$

b) $101_2 + 011_2$

c) $1101_2 + 1001_2$

d) $11100_2 + 01001_2$

4) Construct the corresponding truth table and logic circuits.

a) NOT(A) AND B

b) (A XOR B) AND C

Topic 2: Mathematics Review

5) What is the difference between scalars and vectors?

6) Calculate the magnitude and direction of each of the following vectors:

a) $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$

b) $\begin{pmatrix} -1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

7) Calculate the resulting vector for each of the following operations:

a) $\vec{a} = -2 \cdot) \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

b) $\vec{b} = 3 \cdot \begin{pmatrix} -1 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

c) $\vec{c} = -1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{8} \cdot \begin{pmatrix} -8 \\ 16 \end{pmatrix}$

8) A boat travels north with a speed of 20 knots. There is a wind blowing **from Northwest to Southeast** that has a **magnitude** of $5\sqrt{2}$ knots. What is the resulting velocity $\vec{v}$ of the boat? (state both the magnitude and direction of $\vec{v}$).

9) Express the following complex numbers in polar form using Euler's formula: (express your angles in radians).

a) $4 + 2i$

b) $3 - 3i$

c) $-7i$

10) What is the complex conjugate of the following complex numbers?

a) $(3i)$

b) $(-1-5i)$

c) $e^{-i\frac{\pi}{3}}$

11) Add the following complex numbers:

a) $(2-5i)+(3+5i)$

b) $(1-i)+(4-6i)$

c) $(5)-(4i)$

12) Multiply the following complex numbers:

a) $(2-3i)\cdot(2-3i)$

b) $(1-3i)\cdot(7+2i)$

c) $(4+5i)\cdot(4-5i)$

d) $e^{-i\pi}\cdot e^{i\frac{2\pi}{3}}$

13) Find the modulus of the following complex numbers:

a) $3+3i$

b) $7-5i$

c) $-3e^{-i\frac{\pi}{4}}$

14) Compute the following inner products:

a) $< \begin{pmatrix}-1\\3\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix} >$

b) $< \begin{pmatrix}1\\-5\end{pmatrix}, \begin{pmatrix}0\\-6\end{pmatrix} >$

c) $< \begin{pmatrix}3\\-4i\end{pmatrix}, \begin{pmatrix}-3\\5i\end{pmatrix} >$

15) Consider the vectors:

$$\vec{a} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

What are the linear combinations of $\vec{a}$ and $\vec{b}$ that form the following vectors:

a) $\begin{pmatrix} -1 \\ 9 \end{pmatrix}$

b) $\begin{pmatrix} 8 \\ -7 \end{pmatrix}$

16) Calculate the conjugate transpose of each of the following matrices:

a) $\begin{pmatrix} -2 & 3+i \\ 1-i & -4 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$

a) $\begin{pmatrix} 3+i & -i \\ 1-i & -4 \end{pmatrix}$

17) Calculate the inverse for the following matrices:

a) $\begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}$

b) $\begin{pmatrix} -3 & 2 \\ 4 & -5 \end{pmatrix}$

18) Evaluate the following expressions that involve matrices:

a) $\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} -5 & i \\ -3i & 4 \end{pmatrix} \begin{pmatrix} 3i \\ i \end{pmatrix}$

19) Assume you are in possession of a fair coin and a fair six-sided dice. Calculate the probability of the following events:

a) Flipping two consecutive Heads.

b) Rolling a 1,2,3 or 6.

20) Let X and Y be independent discrete random variables that can be one of the following values: $\{1, 3, 4, 7\}$. However, the probability for X and Y to take each value is different. The following table shows the corresponding probability distributions for X and Y

v	P(X=v)	P(Y=v)
1	0.2	0.5
3	0.3	0.1
4	0.15	0.1
7	0.35	0.3

a) What is the expectation and variance of X?

b) What is the expectation and variance of Y?

c) Evaluate the probability of P(X=3 and Y=7).

Topic 3: Mathematics for Quantum Mechanics

21)State whether the following statements are **True** or **False**:

a) $\langle\psi|\psi\rangle = 1$ for **all** quantum states

b) A quantum state $|\psi\rangle$ is represented as a **vector in a Hilbert space**.

c) All vectors in a vector space are orthogonal to each other.

22) For a set of vectors $\{|a\rangle, |b\rangle, |c\rangle\}$, what is span($\{|a\rangle, |b\rangle, |c\rangle\}$)?

23) Consider a vector space V that contains vectors $|a\rangle, |b\rangle$. Which of the following vectors are also members of V. State **True** or **False** for each.

a) $|x\rangle = 4|a\rangle - 2|b\rangle$

b) $|y\rangle$ for which $\langle a|y\rangle = \langle b|y\rangle = 0$

c) $|z\rangle$ for which $|z\rangle = \alpha|a\rangle + \beta|b\rangle$ has **no solution**

d) $|w\rangle$ for which $|w\rangle = \alpha|a\rangle + \beta|b\rangle$ has a solution.

24) Given: $|A\rangle = \begin{pmatrix} 4 \\ -3e^{-i\frac{\pi}{4}} \end{pmatrix}$. What is $\langle A|$?

25) Given: $M = \begin{pmatrix} 4 & 2i \\ 3 & 2 \end{pmatrix}$, what is $M^\dagger$?

26) What is $|0\rangle$ in vector form?

27) What is $|1\rangle$ in vector form?

28) For a state $|\psi\rangle = \sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}e^{i\frac{\pi}{2}}|1\rangle$, what is the probability that the $|\psi\rangle$ is measured in the state $|1\rangle$?

Important Operators and Superposition States

Let's take some time to review some of the very important operators: the *Pauli operators* (σ_x, σ_y, σ_z) and the *Hadamard* operator (H).

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

We will also be using the superposition states:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

29) Using matrix multiplication, calculate the following:

a) $\sigma_x |0\rangle$

b) $\sigma_x |1\rangle$

c) $\sigma_y |0\rangle$

d) $\sigma_y |1\rangle$

e) $\sigma_z |0\rangle$

f) $\sigma_z |1\rangle$

g) $H |0\rangle$

h) $H |1\rangle$

30) Using the results from Question 29, calculate the following:

a) $\sigma_x |+\rangle$

b) $\sigma_y |-\rangle$

c) $\sigma_z |-\rangle$

d) $H |-\rangle$

e) $H\sigma_z |-\rangle$

Challenge Problems

Bloch Sphere

So far we've talked about complex numbers and vectors. It turns out that in quantum mechanics, quantum states are described by vectors with complex components. The state of a qubit in a quantum computer is given by a two dimensional vector of the form

$$\vec{v} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

These qubit states live on the surface of the **Bloch Sphere**, where θ is the latitudinal angle (ranging from 0 to π) and ϕ is the azimuth angle (ranging from 0 to 2π).

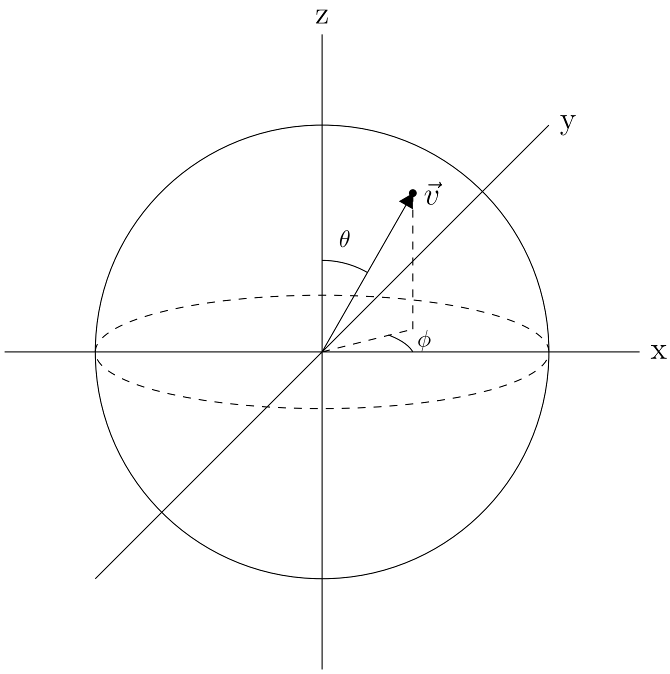


Figure 1: The Bloch sphere showing a general Bloch vector with coordinates described by θ and ϕ.

Understanding the Bloch sphere is at the heart of manipulating quantum states in quantum computing, so let's take some time to practice with Bloch vectors.

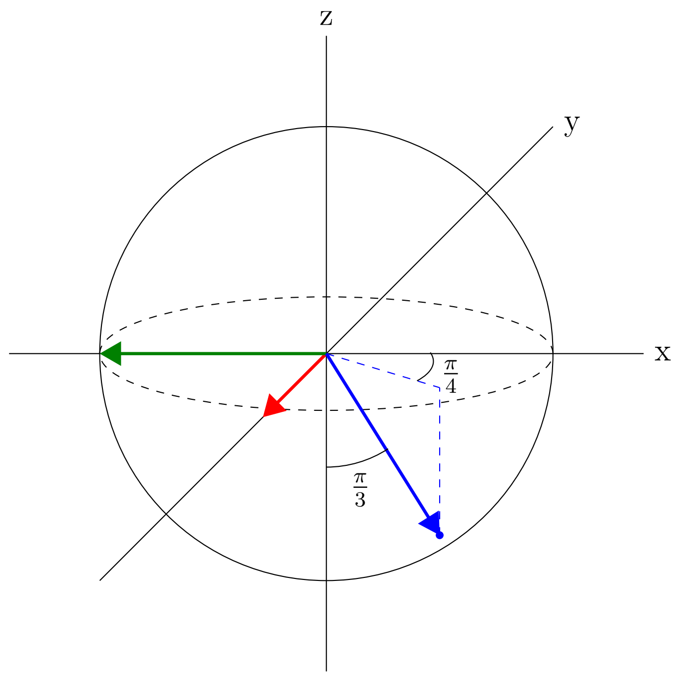
a) Where on the Bloch sphere do these vectors point?

i) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

ii) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

iii) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$

b) Write the vector form of the red, green and blue vectors on the following Bloch sphere.



Rotation Matrices

We will spend a lot of time talking about rotation matrices. As we will see later on, rotation matrices are extremely important in quantum mechanics, so let's take some time to study in detail how they work. As a reminder, a rotation matrix in 2 dimensions takes the form:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

When applied to a vector, this rotation matrix has the action of rotating the vector counter-clockwise by an angle θ. Rotation matrices have some important properties, some of which we will explore here.

a) Because R simply rotates a vector, it should not change it's magnitude. Show that:

$$|R(\theta)\vec{v}| = |\vec{v}|$$

Hint: Here are some steps to help with this

1. Start with a vector with variable elements: $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$
2. Calculate $R(\theta)\vec{v}$ in terms of x and y
3. Calculate the magnitude of both sides (recall: $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$)

It makes sense that if we rotate by an angle θ, then subsequently rotate by $-\theta$, we should be back to where we started. This can be stated in an equation

$$R(\theta)R(-\theta)\vec{v} = \vec{v}$$

Which implies

$$\begin{aligned} R(\theta)R(-\theta) &= I \\ R(-\theta) &= R^{-1}(\theta) \end{aligned} \tag{1}$$

b) Explicitly show that Eq.1 is true. (Recall: I is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$)

Rotations also add to each other. If we rotate a vector by an angle θ_1, then subsequently rotate by angle θ_2, the total angle we have rotated the vector is $\theta_1 + \theta_2$. Our rotation matrix should properly reflect this.

c) Prove the equality:

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$$

Matrix Exponential

We've learned about adding, multiplying, and even taking the inverse of a matrix. But how would one evaluate

$$e^M$$

for some **square matrix** M. At first glance, it doesn't seem to make much sense. However, there is a way to properly define this type of matrix operation, and it comes in very handy in quantum computation.

The definition of the matrix exponential e^M comes from the series definition of the exponential function. Any exponential function $f(x) = e^x$ can be represented by the series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + ...$$

Where $n! = n \times (n-1) \times (n-2) \times ... \times 1$ is the factorial operation.

Using this understanding of the exponential, the matrix exponential e^M is defined to be

$$e^M \equiv \sum_{n=0}^{\infty} \frac{1}{n!} M^n = I + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + ...$$

There are a myriad of interesting properties of matrix exponentials, but in this problem we will focus on a specific case.

Matrix Exponential in Quantum

Consider a 2x2 matrix σ_y with elements:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

a) Verify that

$$\sigma_y^2 = I$$

where as usual I is the identity matrix.

b) Show that

$$e^{i\frac{\theta}{2}\sigma_y} = I\cos\left(\frac{\theta}{2}\right) - i\sigma_y \sin\left(\frac{\theta}{2}\right)$$

Hint: the series expansion for sin and cos may be helpful

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + ... \qquad \sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + ...$$

c) Write $e^{i\frac{\theta}{2}\sigma_y}$ in matrix form (i.e. in the form $e^{i\frac{\theta}{2}\sigma_y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$).

This result should look awfully similar to a rotation matrix; and that's because it is a rotation matrix! Matrices that take the form $e^{i\frac{\theta}{2}\sigma}$ actually perform rotations on Bloch vectors. $e^{i\frac{\theta}{2}\sigma_y}$ specifically describes a rotation about the y-axis on the Bloch sphere by angle θ

$$R_y(\theta) = e^{i\frac{\theta}{2}\sigma_y}$$

There are other matrices σ_x and σ_z that do the same for x and z axes that we will learn more about.

d) Consider the Bloch vector:

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

i) In what direction on the Bloch sphere does this vector point?

ii) What is the matrix we should construct to rotate this vector about the y-axis by $\frac{\pi}{2}$?

iii) Perform the operation, what is the resulting vector?

Solutions

1)

a) 2

b) 7

c) 9

d) 25

2)

a) 101_2

b) 10111_2

c) 1111111_2

3)

a) 110_2

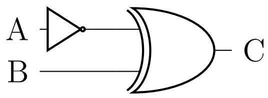
b) 1000_2

c) 10110_2

d) 100101_2

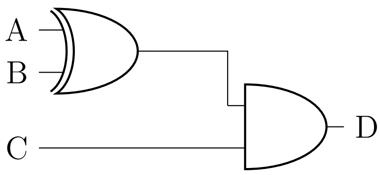
4)Part (a):

A	B	C
0	0	1
0	1	0
1	0	0
1	1	1



Part (b):

A	B	C	D
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0



5) Vectors have a magnitude **and** direction. Scalars only have a magnitude.

6)

a) mag: $\sqrt{34}$, $\theta = \arctan\left(\frac{5}{3}\right)$

b) mag: $\sqrt{\frac{4}{3}}$, $\theta = \frac{2\pi}{3}$

7)

a) $\vec{a} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}$

b) $\vec{b} = \begin{pmatrix} -1 \\ -11 \end{pmatrix}$

c) $\vec{c} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

8) $|\vec{v}| = 5\sqrt{10}$, $\theta = \arctan(3)$

9)

a) $\sqrt{20}e^{i\arctan\left(\frac{1}{2}\right)}$

b) $3\sqrt{2}e^{-i\frac{\pi}{4}}$

c) $7e^{-i\frac{\pi}{2}}$

10)

a) $-3i$

b) $-1 + 5i$

c) $e^{i\frac{\pi}{3}}$

11)

a) 5

b) $5 - 7i$

c) $5 - 4i$

12)

a) $-5 - 12i$

b) $13 - 19i$

c) 41

d) $e^{-i\frac{\pi}{3}}$

13)

a) $3\sqrt{2}$

b) $\sqrt{74}$

c) 3

14)

a) 10

b) 30

c) -29

15)

a) $\begin{pmatrix} -1 \\ 9 \end{pmatrix} \vec{a} + \vec{b}$

b) $\begin{pmatrix} 8 \\ -7 \end{pmatrix} = -3\vec{a} + 2\vec{b}$

16)

a) $\begin{pmatrix} -2 & 1+i \\ 3-i & -4 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 6 \\ 0 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 3-i & 1+i \\ i & -4 \end{pmatrix}$

17)

a) $-\frac{1}{7}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

b) $-\frac{1}{7}\begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}$

18)

a) $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$

b) $3i\begin{pmatrix} -5+i \\ 4-3i \end{pmatrix}$

19)

a) $\frac{1}{4}$

b) $\frac{2}{3}$

20)

a) Ex[X]=4.15, var[X]=5.23

b) Ex[Y]=3.3, var[Y]=6.81

c) P(X=3,Y=7)=0.09

21)

a) True

b) True

c) False

22) span($\{|a\rangle, |b\rangle, |c\rangle$) is the set of all vectors that are linear combinations of $\{|a\rangle, |b\rangle, |c\rangle\}$.

23)

a) True

b) False

c) False

d) True

24) $\langle A| = \begin{pmatrix} 4 & -3e^{i\frac{\pi}{4}} \end{pmatrix}$

25) $M^\dagger = \begin{pmatrix} 4 & 3 \\ -2i & 2 \end{pmatrix}$

26) $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

27) $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

28) $\frac{2}{5}$ 29)

a) $\sigma_x |0\rangle = |1\rangle$

b) $\sigma_x |1\rangle = |0\rangle$

c) $\sigma_y |0\rangle = i |1\rangle$

d) $\sigma_y |1\rangle = -i |0\rangle$

e) $\sigma_z |0\rangle = |0\rangle$

f) $\sigma_z |1\rangle = - |1\rangle$

g) $H |0\rangle = |+\rangle$

h) $H |1\rangle = |-\rangle$

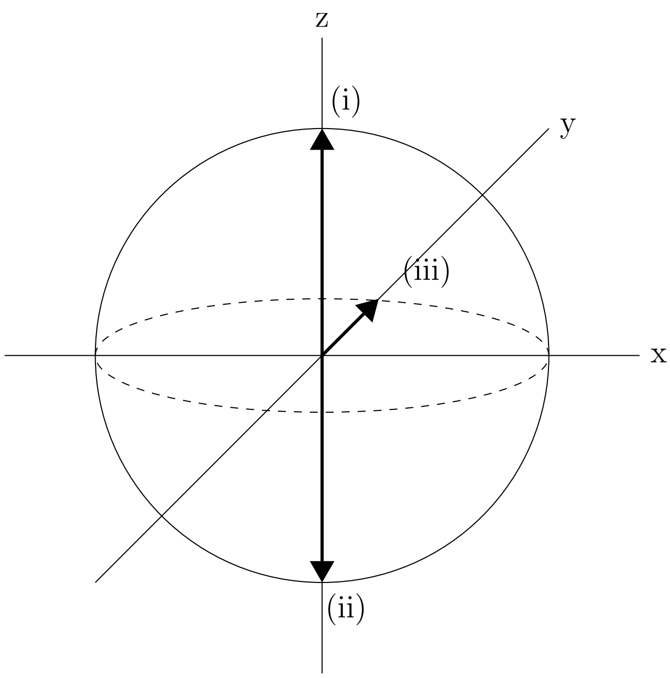
30)

1. $|+\rangle$
2. $- |-\rangle$
3. $|+\rangle$
4. $|1\rangle$
5. $|0\rangle$

Solution to Challenge Problems

Bloch Sphere

Part (a): Bloch vectors (i), (ii) and (iii):



Part (b):

- Red: $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}$
- Green: $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$
- Blue: be careful! The angle θ is always measured with respect to the positive z axis, so $\theta = \pi - \frac{pi}{3} = \frac{2\pi}{3}$ for this one. The vector is $\begin{pmatrix} \frac{1}{2} \\ e^{-i\frac{\pi}{4}} \frac{\sqrt{3}}{2} \end{pmatrix}$

Rotation Matrices

Part (a): We show that the equation holds by applying it to a vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

First, let us calculate $|\vec{v}|^2$.

$$|\vec{v}|^2 = x^2 + y^2$$

Now we apply R to $\vec{v}$

$$R(\theta)\vec{v} = R(\theta)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$R(\theta)\vec{v} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

Then we calculate the magnitude squared of this vector:

$$|R(\theta)\vec{v}|^2 = (R(\theta)\vec{v}) \cdot (R(\theta)\vec{v})$$

$$= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} \cdot \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

$$|R(\theta)\vec{v}|^2 = (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2$$

Now we distribute to find the square of each of these binomials. Be careful, as it's easy to make mistakes here.

$$\begin{aligned} |R(\theta)\vec{v}|^2 = (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 &= x^2\cos^2\theta + y^2\sin^2\theta - 2xy\sin\theta\cos\theta \\ &+ x^2\sin^2\theta + y^2\cos^2\theta + 2xy\sin\theta\cos\theta \end{aligned}$$

$$|R(\theta)\vec{v}|^2 = x^2\cos^2\theta + x^2\sin^2\theta + y^2\cos^2\theta + y^2\sin^2\theta$$

Now we can factor x^2 from the terms with x^2 and y^2 from the terms with y^2.

$$x^2\cos^2\theta + x^2\sin^2\theta + y^2\cos^2\theta + y^2\sin^2\theta = x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta)$$

And finally we can use the trig identity: $\sin^2\theta + \cos^2\theta = 1$

$$|R(\theta)\vec{v}|^2 = x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta) = x^2 + y^2$$

Which we see is exactly the same as $|\vec{v}|^2$, so mission accomplished!

Part (b): First we write what $R(-\theta)$ is, which is:

$$R(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

We need to use the rules for even and odd functions. sin is an odd function and cos is an even function so:

$$\cos(-\theta) = \cos\theta \qquad\qquad \sin(-\theta) = -\sin\theta$$

Using this we simplify

$$R(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Now we do matrix multiplication with $R(\theta)$ to verify the equality.

$$R(\theta)R(-\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Part (c): Here we also explicitly do matrix multiplication

$$R(\theta_1)R(\theta_2) = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1) \\ \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{pmatrix}$$

To simplify this we use the trigonometric identites:

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 \qquad\qquad \cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

And we confirm that

$$R(\theta_1)R(\theta_2) = \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix} = R(\theta_1 + \theta_2)$$

This is also consistent with Eq.1. If we set $\theta_1 = \theta$ and $\theta_2 = -\theta$ we see that:

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2) = R(\theta - \theta) = R(0)$$

A quick check will reveal that indeed $R(0) = I$, which is exactly what we expect.

Matrix Exponential

Part (a):

$$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Part (b):
Using the series expansion for the matrix exponential:

$$e^{-i\frac{\theta}{2}\sigma_y} = \sum_{n=0}^{\infty} \frac{1}{n!}(-i\frac{\theta}{2}\sigma_y)^n$$

$\sigma_y^2 = I$, so $\sigma_y^n = I$ if n is even, and $\sigma_y^n = \sigma_y$ if n is odd. Using this, we write the series as

$$e^{-i\frac{\theta}{2}\sigma_y} = I - i\sigma_y\frac{\theta}{2} - I\frac{1}{2}\left(\frac{\theta}{2}\right)^2 + i\sigma_y\frac{1}{6}\left(\frac{\theta}{2}\right)^3 + I\frac{1}{24}\left(\frac{\theta}{2}\right)^4 - i\sigma_y\frac{1}{120}\left(\frac{\theta}{2}\right)^5 + ...$$

Separating terms that are multiplied by I and $i\sigma_y$ then factoring gives us

$$e^{-i\frac{\theta}{2}\sigma_y} = I(1 - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 + \frac{1}{24}\left(\frac{\theta}{2}\right)^4 + ...) - i\sigma_y(\frac{\theta}{2} - \frac{1}{6}\left(\frac{\theta}{2}\right)^3 + \frac{1}{120}\left(\frac{\theta}{2}\right)^5 + ...)$$

The series in the parentheses multiplied by I and $i\sigma_y$ are $\cos\left(\frac{\theta}{2}\right)$ and $\sin\left(\frac{\theta}{2}\right)$ respectively. This leaves us with the desired result

$$e^{-i\frac{\theta}{2}\sigma_y} = I\cos\left(\frac{\theta}{2}\right) - i\sigma_y\sin\left(\frac{\theta}{2}\right)$$

Part (c):

$$e^{-i\frac{\theta}{2}\sigma_y} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

Part (d):

i) $\vec{v}$ points along the positive z-axis.

ii) $e^{-i\frac{\pi}{4}\sigma_y} = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

iii) $e^{-i\frac{\pi}{4}\sigma_y}\vec{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, this vector represents a superposition state!